

The Meaning of Einstein's Equation

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Abstract

This is a brief introduction to general relativity, designed for both students and teachers of the subject. While there are many excellent expositions of general relativity, few adequately explain the geometrical meaning of the basic equation of the theory: Einstein's equation. Here we give a simple formulation of this equation in terms of the motion of freely falling test particles. We also sketch some of its consequences, and explain how the formulation given here is equivalent to the usual one in terms of tensors. Finally, we include an annotated bibliography of books, articles and websites suitable for the student of relativity.

1 Introduction

General relativity explains gravity as the curvature of spacetime. It's all about geometry. The basic equation of general relativity is called Einstein's equation. In units where $c = 8\pi G = 1$, it says

$$G_{\alpha\beta} = T_{\alpha\beta}.$$

It looks simple, but what does it mean? Unfortunately, the beautiful geometrical meaning of this equation is a bit hard to find in most treatments of relativity. There are many nice popularizations that explain the philosophy behind relativity and the idea of curved spacetime, but most of them don't get around to explaining Einstein's equation and showing how to work out its consequences. There are also more technical introductions which explain Einstein's equation in detail — but here the geometry is often hidden under piles of tensor calculus.

This is a pity, because in fact there is an easy way to express the whole content of Einstein's equation in plain English. In what follows, we start by outlining some differences between special and general relativity. Next we give a verbal formulation of Einstein's equation. Then we derive a few of its consequences, and finally we explain why it is equivalent to the usual formulation in terms of tensors. This article is mainly aimed at those who teach relativity, but except for the last section, we have tried to make it accessible to students, as a sketch of how the subject might be introduced.

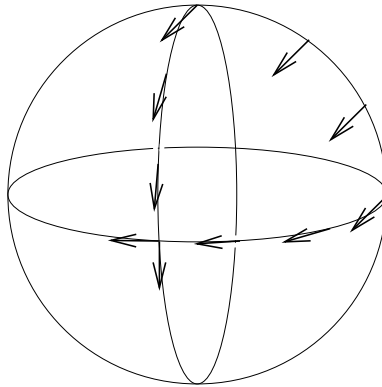
2 Preliminaries

Before stating Einstein's equation, we need a little preparation. We assume the reader is somewhat familiar with special relativity — otherwise general relativity will be too hard. But there are some big differences between special and general relativity, which can cause immense confusion if neglected.

In special relativity, we cannot talk about *absolute* velocities, but only *relative* velocities. For example, we cannot sensibly ask if a particle is at rest, only whether it is at rest relative to another. The reason is that in this theory, velocities are described as vectors in 4-dimensional spacetime, and switching to a different inertial coordinate system can change which way these vectors point, but not whether two of them point the same way.

In general relativity, we cannot even talk about *relative* velocities, except for two particles at the same point of spacetime — that is, at the same place at the same instant. The reason is that in general relativity, we take very seriously the notion that a vector is a little arrow sitting at a particular point in spacetime. To compare vectors at different points of spacetime, we must carry one over to the other. The process of carrying a vector along a path without turning or stretching it is called 'parallel transport'. When spacetime is curved, the result of parallel transport from one point to another depends on the path taken! In fact, this is the very definition of what it means for spacetime to be curved. Thus it is ambiguous to ask whether two particles have the same velocity vector unless they are at the same point of spacetime.

It is hard to imagine the curvature of 4-dimensional spacetime, but it is easy to see it in a 2-dimensional surface, like a sphere. The sphere fits nicely in 3-dimensional flat Euclidean space, so we can visualize vectors on the sphere as 'tangent vectors'. If we parallel transport a tangent vector from the north pole to the equator by going straight down a meridian, we get a different result than if we go down another meridian and then along the equator:



Because of this analogy, in general relativity vectors are usually called 'tangent vectors'. However, it is important not to take this analogy too seriously. Our curved spacetime need not be embedded in some higher-dimensional flat spacetime for us

to understand its curvature, or the concept of tangent vector. The mathematics of tensor calculus is designed to let us handle these concepts ‘intrinsically’ — i.e., working solely within the 4-dimensional spacetime in which we find ourselves. This is one reason tensor calculus is so important in general relativity.

Now, in special relativity we can think of an inertial coordinate system, or ‘inertial frame’, as being defined by a field of clocks, all at rest relative to each other. In general relativity this makes no sense, since we can only unambiguously define the relative velocity of two clocks if they are at the same location. Thus the concept of inertial frame, so important in special relativity, is *banned* from general relativity!

If we are willing to put up with limited accuracy, we can still talk about the relative velocity of two particles in the limit where they are very close, since curvature effects will then be very small. Using this, we can approximately talk about a ‘local’ inertial coordinate system. However, one must remember that this notion only makes perfect sense in the limit where the region of spacetime covered by the coordinate system goes to zero in size.

Einstein’s equation can be expressed as a statement about the relative acceleration of very close test particles in free fall. Let us clarify these terms a bit. A ‘test particle’ is an idealized point particle with energy and momentum so small that its effects on spacetime curvature are negligible. A particle is said to be in ‘free fall’ when its motion is affected by no forces except gravity. In general relativity, a test particle in free fall will trace out a ‘geodesic’. This means that its velocity vector is parallel transported along the curve it traces out in spacetime. A geodesic is the closest thing there is to a straight line in curved spacetime.

Again, all this is easier to visualize in 2d space rather than 4d spacetime. A person walking on a sphere ‘following their nose’ will trace out a geodesic — that is, a great circle. Suppose two people stand side-by-side on the equator and start walking north, both following geodesics. Though they start out walking parallel to each other, the distance between them will gradually start to shrink, until finally they bump into each other at the north pole. If they didn’t understand the curved geometry of the sphere, they might think a ‘force’ was pulling them together.

Similarly, in general relativity gravity is not really a ‘force’, but just a manifestation of the curvature of spacetime. Note: not the curvature of space, but of *spacetime*. The distinction is crucial. If you toss a ball, it follows a parabolic path. This is far from being a geodesic in *space*: space is curved by the Earth’s gravitational field, but it is certainly not so curved as all that! The point is that while the ball moves a short distance in space, it moves an enormous distance in *time*, since one second equals about 300,000 kilometers in units where $c = 1$. This allows a slight amount of spacetime curvature to have a noticeable effect.

3 Einstein's Equation

To state Einstein's equation in simple English, we need to consider a round ball of test particles that are initially 'comoving' — i.e., at rest relative to each other. As we have seen, this is a sensible notion only in the limit where the ball is very small. If we start with a very small ball of comoving particles, it will, to second order in time, become an ellipsoid as time passes. This should not be too surprising, because any linear transformation applied to a ball gives an ellipsoid, and as the saying goes, "everything is linear to first order". Here we get a bit more: the relative velocity of the particles starts out being zero, so to first order in time the ball does not change shape at all: the change is a second-order effect.

Let $V(t)$ be the volume of the ball after a proper time t has elapsed, as measured by the particle at the center of the ball. Then Einstein's equation says:

$$\frac{\ddot{V}}{V}\Big|_{t=0} = -\frac{1}{2} \begin{pmatrix} \text{flow of } t\text{-momentum in } t \text{ direction} + \\ \text{flow of } x\text{-momentum in } x \text{ direction} + \\ \text{flow of } y\text{-momentum in } y \text{ direction} + \\ \text{flow of } z\text{-momentum in } z \text{ direction} \end{pmatrix}$$

where these flows are measured at the center of the ball at time zero, using local inertial coordinates. These flows are the diagonal components of a 4×4 matrix T called the 'stress-energy tensor'. The components $T_{\alpha\beta}$ of this matrix say how much momentum in the α direction is flowing in the β direction through a given point of spacetime, where $\alpha, \beta = t, x, y, z$. The flow of t -momentum in the t -direction is just the energy density, often denoted ρ . The flow of x -momentum in the x -direction is the 'pressure in the x direction' denoted P_x , and similarly for y and z . It takes a while to figure out why pressure is really the flow of momentum, but it is eminently worth doing. Most texts explain this fact by considering the example of an ideal gas.

In any event, we may summarize Einstein's equation as follows:

$$\frac{\ddot{V}}{V}\Big|_{t=0} = -\frac{1}{2}(\rho + P_x + P_y + P_z). \quad (1)$$

This equation says that positive energy density and positive pressure curve spacetime in a way that makes a freely falling ball of point particles tend to shrink. Since $E = mc^2$ and we are working in units where $c = 1$, ordinary mass density counts as a form of energy density. Thus a massive object will make a swarm of freely falling particles at rest around it start to shrink. In short: *gravity attracts*.

We promised to state Einstein's equation in plain English, but have not done so yet. Here it is:

The rate at which a small initially comoving ball of freely falling test particles begins to shrink is proportional to its volume times: the energy density at the center of the ball, plus the pressure in the x direction at that point, plus the pressure in the y direction, plus the pressure in the z direction.

4 Some Consequences

The formulation of Einstein's equation we have given is certainly not the best for most applications of general relativity. For example, in 1915 Einstein used general relativity to correctly compute the anomalous precession of the orbit of Mercury and also the deflection of starlight by the Sun's gravitational field. Both these calculations would be very hard starting from equation (1); they really call for the full apparatus of tensor calculus. However, we can easily use our formulation of Einstein's equation to get a qualitative — and sometimes even quantitative — understanding of *some* consequences of general relativity. We have already seen that it explains how gravity attracts. We sketch a few other consequences below.

Tidal Forces, Gravitational Waves

Let $V(t)$ be the volume of a small ball of initially comoving test particles in free fall. In the vacuum there is no energy density or pressure, so $\ddot{V}|_{t=0} = 0$, but the curvature of spacetime can still distort the ball. For example, suppose you drop a small ball of instant coffee when making coffee in the morning. The grains of coffee closer to the earth accelerate towards it a bit more, causing the ball to start stretching in the vertical direction. However, as the grains all accelerate towards the center of the earth, the ball also starts being squashed in the two horizontal directions. Einstein's equation says that if we treat the coffee grains as test particles, these two effects cancel each other when we calculate the second derivative of the ball's volume, leaving us with $\ddot{V}|_{t=0} = 0$. It is a fun exercise to check this using Newton's theory of gravity!

This stretching/squashing of a ball of falling coffee grains is an example of what people call 'tidal forces'. As the name suggests, another example is the tendency for the ocean to be stretched in one direction and squashed in the other two by the gravitational pull of the moon.

Gravitational waves are another example of how spacetime can be curved even in the vacuum. General relativity predicts that when any heavy object wiggles, it sends out ripples of spacetime curvature which propagate at the speed of light. This is far from obvious starting from our formulation of Einstein's equation! It also predicts that as one of these ripples of curvature passes by, a small ball of initially comoving test particles will be stretched in one transverse direction while being squashed in the other transverse direction. From what we have already said, these effects must precisely cancel when we compute $\ddot{V}|_{t=0}$.

Hulse and Taylor won the Nobel prize in 1993 for careful observations of a binary neutron star which is slowly spiralling down, just as general relativity predicts it should, as it loses energy by emitting gravitational radiation. Gravitational waves have not been *directly* observed, but there are a number of projects underway to detect them. For example, the LIGO project will bounce a laser between hanging mirrors in an L-shaped detector, to see how one leg of the detector is stretched while

the other is squashed. Both legs are 4 kilometers long, and the detector is designed to be sensitive to a 10^{-18} -meter change in length of the arms.

Gravitational Collapse

One remarkable feature of this equation is the pressure term: it says that not only energy density but also pressure causes gravitational attraction. This may seem to violate our intuition that pressure makes matter want to expand! Here, however, we are talking about *gravitational* effects of pressure, which are undetectably small in everyday circumstances. This is clear if we restore the units:

$$\left. \frac{\ddot{V}}{V} \right|_{t=0} = -\frac{4\pi G}{c^4}(\rho + P_x + P_y + P_z),$$

since on the human scale G is very small and c is very big.

In everyday situations the energy density ρ vastly exceeds the pressure terms, since to convert from mass density to energy density we multiply by c^2 . However, the gravitational effects of pressure become very important near the end of the life of a large star. A neutron star is held up by the degeneracy pressure of the neutronium it consists of, but above a mass of about 2 solar masses a nonrotating neutron star will inevitably collapse to form a black hole, thanks in part to the gravitational attraction caused by pressure.

The Big Bang

Starting from our formulation of Einstein's equation it is easy to understand some basic facts about the big bang cosmology. Let us assume that the universe is very close to being homogeneous and isotropic, at least on large length scales. We can approximately treat the galaxies in any ball of space as a 'small ball of test particles,' and

$$\frac{\ddot{V}}{V} = -\frac{1}{2}(\rho + P_x + P_y + P_z)$$

where $V(t)$ is the volume of this ball. We have seen this equation when $t = 0$ and the ball starts out comoving, but in this highly symmetric situation it actually holds for all times, even though the galaxies are moving apart! Moreover, since the universe is isotropic, the various components of pressure must be equal: $P_x = P_y = P_z = P$. Thus we have

$$\frac{\ddot{V}}{V} = -\frac{1}{2}(\rho + 3P).$$

Isotropy also implies that our ball of galaxies will remain round as time passes, so $V \propto R^3$ where R is the ball's radius, giving

$$\frac{3\ddot{R}}{R} = -\frac{1}{2}(\rho + 3P). \quad (2)$$

To go further, we must make more assumptions about the nature of the matter comprising the universe. At present the pressure is very low, so we may approximately assume $P = 0$, obtaining

$$\frac{3\ddot{R}}{R} = -\frac{\rho}{2}.$$

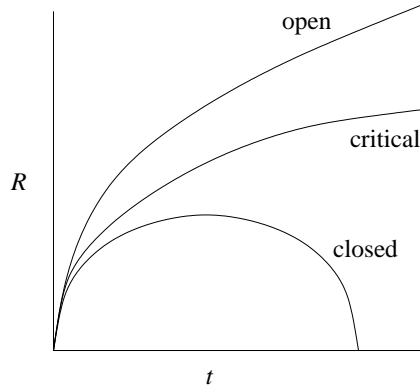
If the energy density of the universe is mainly due to the mass in galaxies, and we model the galaxies as particles in free fall, ‘conservation of galaxies’ implies that $\rho R^3 = c$ for some constant c . This gives

$$\frac{3\ddot{R}}{R} = -\frac{c}{2R^3}$$

or

$$\ddot{R} = -\frac{c}{6R^2}.$$

Amusingly, this is the same as the equation of motion for a particle in an attractive $1/R^2$ force field. In other words, the equation governing this simplified cosmology is the same as the Newtonian equation for what happens when you throw a ball vertically upwards from the earth! This is a nice example of the unity of physics. Since “whatever goes up must come down — unless it exceeds escape velocity,” the solutions of this equation look roughly like this:



In other words, the universe started out with a big bang! It will expand forever if its current rate of expansion is sufficiently high compared to its current density, but it will recollapse in a ‘big crunch’ otherwise.

The Cosmological Constant

The simplified big bang model just described is certainly inaccurate for the very early history of the universe, when the pressure of radiation was important. However, recent observations seem to indicate that it is seriously inaccurate even in the present epoch. First of all, it seems that much of the energy density is not accounted for by known forms of matter. Still more shocking, it seems that the expansion of the universe may be accelerating rather than slowing down! One possibility is that the

energy density and pressure are nonzero even for the vacuum. For the vacuum to not pick out a preferred notion of ‘rest’, its stress energy tensor must be proportional to the metric. In local inertial coordinates this means

$$T = \begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & -\Lambda & 0 & 0 \\ 0 & 0 & -\Lambda & 0 \\ 0 & 0 & 0 & -\Lambda \end{pmatrix}$$

where Λ is called the ‘cosmological constant’. This amounts to giving empty space an energy density equal to Λ and pressure equal to $-\Lambda$, so that $\rho + 3P$ for the vacuum is -2Λ . Here pressure effects dominate because there are more dimensions of space than of time! If we add this cosmological constant term to equation (2), we get

$$\frac{3\ddot{R}}{R} = -\frac{1}{2}(\rho + 3P - 2\Lambda),$$

where ρ and P are the energy density and pressure due to matter. If we treat matter as we did before, this gives

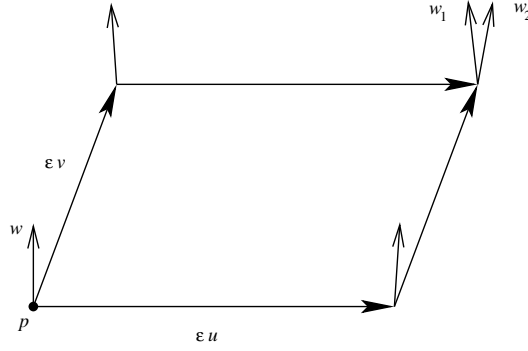
$$\frac{3\ddot{R}}{R} = -\frac{c}{2R^3} + \Lambda.$$

Thus, once the universe expands sufficiently, the cosmological constant becomes more important than the energy density of matter when it comes to determining the fate of the universe. If $\Lambda > 0$, a roughly exponential expansion will then ensue.

5 The Mathematical Details

To see why equation (1) is equivalent to the usual formulation of Einstein’s equation, we need a bit of tensor calculus. In particular, we need to understand the Riemann curvature tensor and the geodesic deviation equation. For a detailed explanation of these, the reader must turn to some of the texts in the bibliography. Here we briefly sketch the main ideas.

When spacetime is curved, the result of parallel transport depends on the path taken. To quantify this notion, pick two vectors u and v at a point p in spacetime. In the limit where $\epsilon \rightarrow 0$, we can approximately speak of a ‘parallelogram’ with sides ϵu and ϵv . Take another vector w at p and parallel transport it first along ϵv and then along ϵu to the opposite corner of this parallelogram. The result is some vector w_1 . Alternatively, parallel transport w first along ϵv and then along ϵu . The result is a slightly different vector, w_2 :



The limit

$$\lim_{\epsilon \rightarrow 0} \frac{w_2 - w_1}{\epsilon^2} = R(u, v)w \quad (3)$$

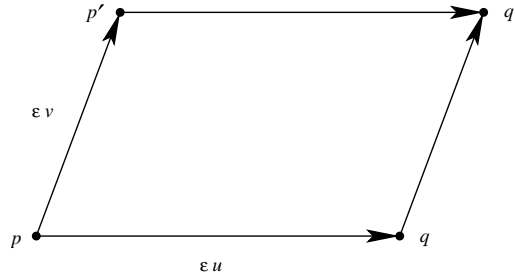
is well-defined, and it measures the curvature of spacetime at the point p . In local coordinates we can write it as

$$R(u, v)w = R_{\beta\gamma\delta}^{\alpha} u^{\beta} v^{\gamma} w^{\delta},$$

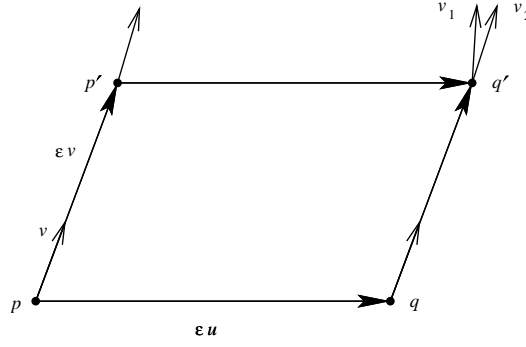
where as usual we sum over repeated indices. The quantity $R_{\beta\gamma\delta}^{\alpha}$ is called the ‘Riemann curvature tensor’.

We can use this tensor to see how nearby particles in free fall will accelerate relative to one another if they start out comoving. Consider two freely falling particles at nearby points p and q . Let v be the velocity of the particle at p , and let ϵu be the vector from p to q . Since the two particles are initially comoving, the velocity of the particle at q is obtained by parallel transporting v along ϵu .

Now let us wait a short while. Both particles trace out geodesics as time passes, and at time ϵ they will be at new points, say p' and q' . The point p' is displaced from p by an amount ϵv , so we get a little parallelogram, exactly as in the definition of the Riemann curvature:



Next let us compute the new relative velocity of the two particles. To compare vectors we must carry one to another using parallel transport. Let v_1 be the vector we get by taking the velocity vector of the particle at p' and parallel transporting it to q' along the top edge of our parallelogram. Let v_2 be the velocity of the particle at q' . The difference $v_2 - v_1$ is the new relative velocity. Here is a picture of the whole situation:



The vector v is depicted as shorter than ϵv for purely artistic reasons.

It follows that over this passage of time, the average relative acceleration of the two particles is $a = (v_2 - v_1)/\epsilon$. By equation (3),

$$\lim_{\epsilon \rightarrow 0} \frac{v_2 - v_1}{\epsilon^2} = R(u, v)v,$$

so

$$\lim_{\epsilon \rightarrow 0} \frac{a}{\epsilon} = R(u, v)v.$$

This is called the ‘geodesic deviation equation’. From the definition of the Riemann curvature it is easy to see that $R(u, v)w = -R(v, u)w$, so we can also write this equation as

$$\lim_{\epsilon \rightarrow 0} \frac{a^\alpha}{\epsilon} = -R_{\beta\gamma\delta}^\alpha v^\beta u^\gamma v^\delta. \quad (4)$$

Using this equation we can work out the second time derivative of the volume $V(t)$ of a small ball of initially comoving particles. For this we must let u range over an orthonormal basis of tangent vectors, and sum the ‘outwards’ component of acceleration for each one of these. By equation (4) this gives

$$\lim_{V \rightarrow 0} \frac{\ddot{V}}{V} \Big|_{t=0} = -R_{\beta\alpha\delta}^\alpha v^\beta v^\delta.$$

In terms of the ‘Ricci tensor’

$$R_{\beta\delta} = R_{\beta\alpha\delta}^\alpha$$

we may write this as:

$$\lim_{V \rightarrow 0} \frac{\ddot{V}}{V} \Big|_{t=0} = -R_{\beta\delta} v^\beta v^\delta.$$

In local inertial coordinates where the ball starts out at rest we have $v = (1, 0, 0, 0)$, so

$$\lim_{V \rightarrow 0} \frac{\ddot{V}}{V} \Big|_{t=0} = -R_{tt}. \quad (5)$$

In short, the Ricci tensor says how any small freely falling ball of initially comoving test particles starts changing in volume. The Ricci tensor only captures some of the

information in the Riemann curvature tensor. The rest is captured by something called the ‘Weyl tensor’, which says how any such ball starts changing in shape. The Weyl tensor describes tidal forces, gravitational waves and the like.

Now, Einstein’s equation in its usual form says

$$G_{\alpha\beta} = T_{\alpha\beta}.$$

Here the right side is the stress-energy tensor, while the left side, the ‘Einstein tensor’, is just an abbreviation for a quantity constructed from the Ricci tensor:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R^\gamma_\gamma.$$

Thus Einstein’s equation really says

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R^\gamma_\gamma = T_{\alpha\beta}. \quad (6)$$

This implies

$$R^\alpha_\alpha - \frac{1}{2}g^\alpha_\alpha R^\gamma_\gamma = T^\alpha_\alpha,$$

but $g^\alpha_\alpha = 4$, so

$$-R^\alpha_\alpha = T^\alpha_\alpha.$$

Plugging this in equation (6), we get

$$R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T^\gamma_\gamma. \quad (7)$$

This is equivalent version of Einstein’s equation, but with the roles of R and T switched! The good thing about this version is that it gives a formula for the Ricci tensor, which has a simple geometrical meaning.

Equation (7) will be true if any one component holds in all local inertial coordinate systems. This is a bit like the observation that all of Maxwell’s equations are contained in Gauss’ law and $\nabla \cdot B = 0$. Of course, this is only true if we know how the fields transform under change of coordinates. Here we assume that the transformation laws are known. Given this, Einstein’s equation is equivalent to the fact that

$$R_{tt} = T_{tt} - \frac{1}{2}g_{tt}T^\gamma_\gamma \quad (8)$$

in every local inertial coordinate system about every point. In such coordinates we have

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (9)$$

so $g_{tt} = -1$ and

$$T_{\gamma}^{\gamma} = -T_{tt} + T_{xx} + T_{yy} + T_{zz}.$$

Equation (8) thus says that

$$R_{tt} = \frac{1}{2}(T_{tt} + T_{xx} + T_{yy} + T_{zz}).$$

By equation (5), this is equivalent to

$$\lim_{V \rightarrow 0} \frac{\ddot{V}}{V} \Big|_{t=0} = -\frac{1}{2}(T_{tt} + T_{xx} + T_{yy} + T_{zz}),$$

as desired.

Websites

There is a lot of material on general relativity available online. Most of it can be found starting from here:

Relativity on the World Wide Web, C. Hillman,

<http://math.ucr.edu/home/baez/relativity.html>

The beginner will especially enjoy the many gorgeous websites aimed at helping one visualize relativity. There are also books available for free online, such as this:

Lecture Notes on General Relativity, S. M. Carroll,

<http://pancake.uchicago.edu/~carroll/notes/>

Part of learning relativity is working one's way through certain classic confusions. The most common are dealt with here:

Frequently Asked Questions on Relativity, edited by C. Hillman and N. Urban,

<http://math.ucr.edu/home/baez/physics/relativity.html>

Nontechnical Books

Before diving into the details of general relativity, it is good to get oriented by reading some less technical books. Here are three excellent ones written by leading experts on the subject:

General Relativity from A to B, R. Geroch (University of Chicago, Chicago, 1981).

Black Holes and Time Warps: Einstein's Outrageous Legacy, K. S. Thorne (W. W. Norton, New York, 1995).

Space, Time, and Gravity: the Theory of the Big Bang and Black Holes, R. M. Wald (University of Chicago, Chicago, 1992).

Special Relativity

Before delving into general relativity in a more technical way, one must get up to speed on special relativity. Here are two excellent texts for this:

Introduction to Special Relativity, W. Rindler (Oxford University, Oxford 1991).

Spacetime Physics: Introduction to Special Relativity, E. F. Taylor and J. A. Wheeler (W. H. Freeman, New York, 1992).

Introductory Texts

When one is ready to tackle the details of general relativity, it is probably good to start with one of these textbooks:

Introducing Einstein's Relativity, R. A. D'Inverno (Oxford University, Oxford, 1992).

Introduction to General Relativity, L. Hughston and K. P. Tod (Cambridge University, Cambridge, 1991).

A First Course in General Relativity, B. F. Schutz (Cambridge University, Cambridge, 1985).

General Relativity: An Introduction to the Theory of the Gravitational Field, H. Stephani (Cambridge University, Cambridge, 1990).

More Comprehensive Texts

To become an expert on general relativity, one really must tackle these classic texts:

Gravitation, C. W. Misner, K. S. Thorne, and J. A. Wheeler (W. H. Freeman, New York, 1973).

General Relativity, R. M. Wald (University of Chicago, Chicago, 1984).

Differential Geometry

The serious student of general relativity will experience a constant need to learn more tensor calculus — or in modern terminology, ‘differential geometry’. Some of this can be found in the texts listed above, but it is also good to read mathematics texts. Here are a few:

Gauge Fields, Knots and Gravity, J. C. Baez and J. P. Muniain (World Scientific, Singapore, 1994).

An Introduction to Differentiable Manifolds and Riemannian Geometry,
W. M. Boothby (Academic Press, New York, 1986).

Semi-Riemannian Geometry with Applications to Relativity, B. O'Neill
(Academic Press, New York, 1983).